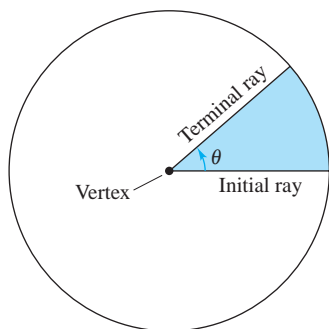


SECTION D.3 Review of Trigonometric Functions

Angles and Degree Measure • Radian Measure • The Trigonometric Functions • Evaluating Trigonometric Functions • Solving Trigonometric Equations • Graphs of Trigonometric Functions

Angles and Degree Measure



Standard position of an angle
Figure D.24

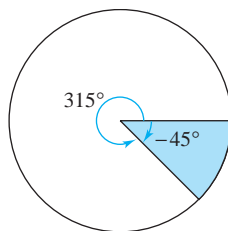
An **angle** has three parts: an **initial ray**, a **terminal ray**, and a **vertex** (the point of intersection of the two rays), as shown in Figure D.24. An angle is in **standard position** if its initial ray coincides with the positive x -axis and its vertex is at the origin. We assume that you are familiar with the degree measure of an angle.* It is common practice to use θ (the Greek lowercase *theta*) to represent both an angle and its measure. Angles between 0° and 90° are **acute**, and angles between 90° and 180° are **obtuse**.

Positive angles are measured *counterclockwise*, and negative angles are measured *clockwise*. For instance, Figure D.25 shows an angle whose measure is -45° . You cannot assign a measure to an angle by simply knowing where its initial and terminal rays are located. To measure an angle, you must also know how the terminal ray was revolved. For example, Figure D.25 shows that the angle measuring -45° has the same terminal ray as the angle measuring 315° . Such angles are **coterminal**. In general, if θ is any angle, then

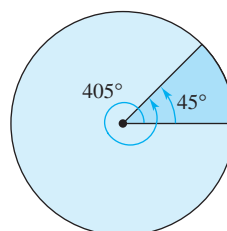
$$\theta + n(360), \quad n \text{ is a nonzero integer}$$

is coterminal with θ .

An angle that is larger than 360° is one whose terminal ray has been revolved more than one full revolution counterclockwise, as shown in Figure D.26. You can form an angle whose measure is less than -360° by revolving a terminal ray more than one full revolution clockwise.



Coterminal angles
Figure D.25



Coterminal angles
Figure D.26

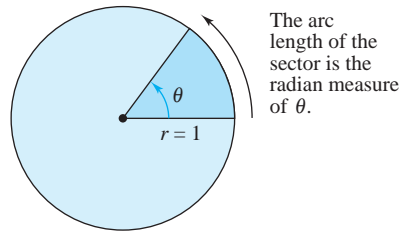
NOTE It is common to use the symbol θ to refer to both an *angle* and its *measure*. For instance, in Figure D.26, you can write the measure of the smaller angle as $\theta = 45^\circ$.

*For a more complete review of trigonometry, see *Precalculus, 5th edition*, by Larson and Hostetler (Boston, Massachusetts: Houghton Mifflin, 2001).

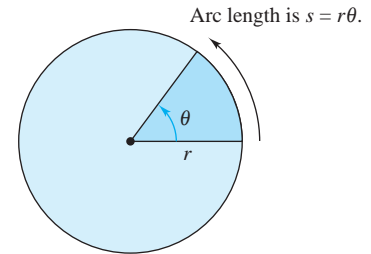
Radian Measure

To assign a radian measure to an angle θ , consider θ to be a central angle of a circle of radius 1, as shown in Figure D.27. The **radian measure** of θ is then defined to be the length of the arc of the sector. Because the circumference of a circle is $2\pi r$, the circumference of a **unit circle** (of radius 1) is 2π . This implies that the radian measure of an angle measuring 360° is 2π . In other words, $360^\circ = 2\pi$ radians.

Using radian measure for θ , the length s of a circular arc of radius r is $s = r\theta$, as shown in Figure D.28.

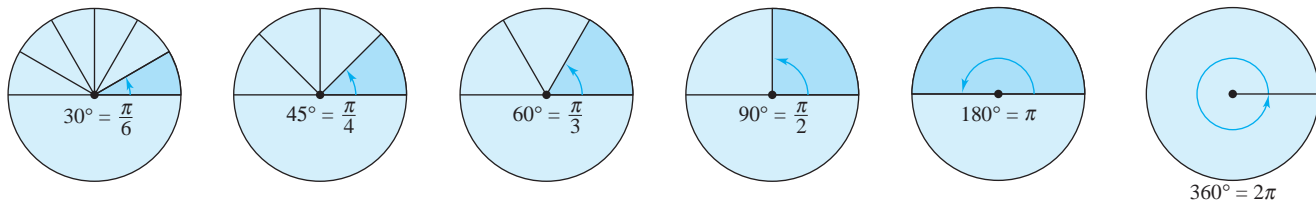


Unit circle
Figure D.27



Circle of radius r
Figure D.28

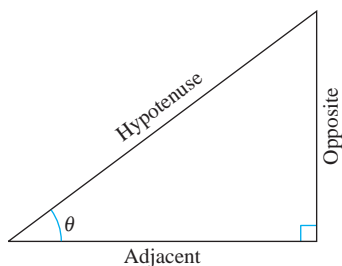
You should know the conversions of the common angles shown in Figure D.29. For other angles, use the fact that 180° is equal to π radians.



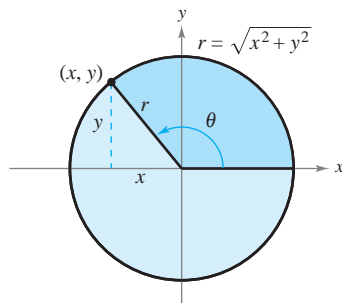
Radian and degree measure for several common angles
Figure D.29

EXAMPLE 1 Conversions Between Degrees and Radians

- $40^\circ = (40 \text{ deg}) \left(\frac{\pi \text{ rad}}{180 \text{ deg}} \right) = \frac{2\pi}{9} \text{ radians}$
- $-270^\circ = (-270 \text{ deg}) \left(\frac{\pi \text{ rad}}{180 \text{ deg}} \right) = -\frac{3\pi}{2} \text{ radians}$
- $-\frac{\pi}{2} \text{ radians} = \left(-\frac{\pi}{2} \text{ rad} \right) \left(\frac{180 \text{ deg}}{\pi \text{ rad}} \right) = -90^\circ$
- $\frac{9\pi}{2} \text{ radians} = \left(\frac{9\pi}{2} \text{ rad} \right) \left(\frac{180 \text{ deg}}{\pi \text{ rad}} \right) = 810^\circ$



Sides of a right triangle
Figure D.30



An angle in standard position
Figure D.31

The Trigonometric Functions

There are two common approaches to the study of trigonometry. In one, the trigonometric functions are defined as ratios of two sides of a right triangle. In the other, these functions are defined in terms of a point on the terminal side of an angle in standard position. We define the six trigonometric functions, **sine**, **cosine**, **tangent**, **cotangent**, **secant**, and **cosecant** (abbreviated as sin, cos, etc.), from both viewpoints.

Definition of the Six Trigonometric Functions

Right triangle definitions, where $0 < \theta < \frac{\pi}{2}$ (see Figure D.30).

$$\begin{aligned} \sin \theta &= \frac{\text{opposite}}{\text{hypotenuse}} & \cos \theta &= \frac{\text{adjacent}}{\text{hypotenuse}} & \tan \theta &= \frac{\text{opposite}}{\text{adjacent}} \\ \csc \theta &= \frac{\text{hypotenuse}}{\text{opposite}} & \sec \theta &= \frac{\text{hypotenuse}}{\text{adjacent}} & \cot \theta &= \frac{\text{adjacent}}{\text{opposite}} \end{aligned}$$

Circular function definitions, where θ is any angle (see Figure D.31).

$$\begin{aligned} \sin \theta &= \frac{y}{r} & \cos \theta &= \frac{x}{r} & \tan \theta &= \frac{y}{x} \\ \csc \theta &= \frac{r}{y} & \sec \theta &= \frac{r}{x} & \cot \theta &= \frac{x}{y} \end{aligned}$$

The following trigonometric identities are direct consequences of the definitions. (ϕ is the Greek letter phi.)

Trigonometric Identities [Note that $\sin^2 \theta$ is used to represent $(\sin \theta)^2$.]

Pythagorean Identities:

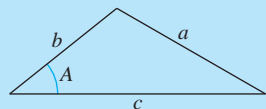
$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \sec^2 \theta \\ \cot^2 \theta + 1 &= \csc^2 \theta \end{aligned}$$

Sum or Difference of Two Angles:

$$\begin{aligned} \sin(\theta \pm \phi) &= \sin \theta \cos \phi \pm \cos \theta \sin \phi \\ \cos(\theta \pm \phi) &= \cos \theta \cos \phi \mp \sin \theta \sin \phi \\ \tan(\theta \pm \phi) &= \frac{\tan \theta \pm \tan \phi}{1 \mp \tan \theta \tan \phi} \end{aligned}$$

Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc \cos A$$



Reduction Formulas:

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \sin \theta &= -\sin(\theta - \pi) \\ \cos(-\theta) &= \cos \theta & \cos \theta &= -\cos(\theta - \pi) \\ \tan(-\theta) &= -\tan \theta & \tan \theta &= \tan(\theta - \pi) \end{aligned}$$

Half-Angle Formulas:

$$\begin{aligned} \sin^2 \theta &= \frac{1}{2}(1 - \cos 2\theta) \\ \cos^2 \theta &= \frac{1}{2}(1 + \cos 2\theta) \end{aligned}$$

Reciprocal Identities:

$$\begin{aligned} \csc \theta &= \frac{1}{\sin \theta} \\ \sec \theta &= \frac{1}{\cos \theta} \\ \cot \theta &= \frac{1}{\tan \theta} \end{aligned}$$

Double-Angle Formulas:

$$\begin{aligned} \sin 2\theta &= 2 \sin \theta \cos \theta \\ \cos 2\theta &= 2 \cos^2 \theta - 1 \\ &= 1 - 2 \sin^2 \theta \\ &= \cos^2 \theta - \sin^2 \theta \end{aligned}$$

Quotient Identities:

$$\begin{aligned} \tan \theta &= \frac{\sin \theta}{\cos \theta} \\ \cot \theta &= \frac{\cos \theta}{\sin \theta} \end{aligned}$$

Evaluating Trigonometric Functions

There are two ways to evaluate trigonometric functions: (1) decimal approximations with a calculator (or a table of trigonometric values) and (2) exact evaluations using trigonometric identities and formulas from geometry. When using a calculator to evaluate a trigonometric function, remember to set the calculator to the appropriate mode—degree mode or radian mode.

EXAMPLE 2 Exact Evaluation of Trigonometric Functions

Evaluate the sine, cosine, and tangent of $\frac{\pi}{3}$.

Solution Begin by drawing the angle $\theta = \frac{\pi}{3}$ in standard position, as shown in Figure D.32. Then, because $60^\circ = \frac{\pi}{3}$ radians, you can draw an equilateral triangle with sides of length 1 and θ as one of its angles. Because the altitude of this triangle bisects its base, you know that $x = \frac{1}{2}$. Using the Pythagorean Theorem, you obtain

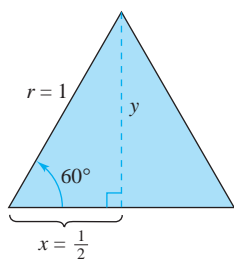
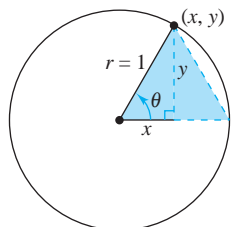
$$y = \sqrt{r^2 - x^2} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

Now, knowing the values of x , y , and r , you can write the following.

$$\sin \frac{\pi}{3} = \frac{y}{r} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$$

$$\cos \frac{\pi}{3} = \frac{x}{r} = \frac{1/2}{1} = \frac{1}{2}$$

$$\tan \frac{\pi}{3} = \frac{y}{x} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$



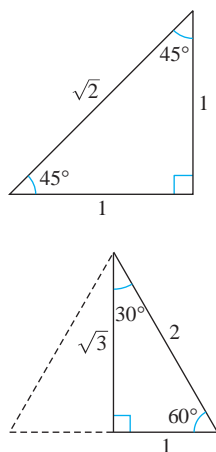
The angle $\frac{\pi}{3}$ in standard position
Figure D.32

NOTE All angles in this text are measured in radians unless stated otherwise. For example, when we write $\sin 3$, we mean the sine of 3 radians, and when we write $\sin 3^\circ$, we mean the sine of 3 degrees.

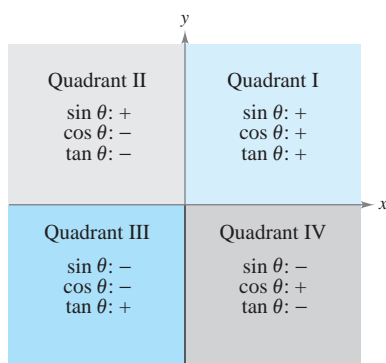
The degree and radian measures of several common angles are given in the table below, along with the corresponding values of the sine, cosine, and tangent (see Figure D.33).

Common First Quadrant Angles

Degrees	0	30°	45°	60°	90°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined



Common angles
Figure D.33



Quadrant signs for trigonometric functions
Figure D.34

The quadrant signs for the sine, cosine, and tangent functions are shown in Figure D.34. To extend the use of the table on the preceding page to angles in quadrants other than the first quadrant, you can use the concept of a **reference angle** (see Figure D.35), with the appropriate quadrant sign. For instance, the reference angle for $3\pi/4$ is $\pi/4$, and because the sine is positive in the second quadrant, you can write

$$\sin \frac{3\pi}{4} = +\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

Similarly, because the reference angle for 330° is 30° , and the tangent is negative in the fourth quadrant, you can write

$$\tan 330^\circ = -\tan 30^\circ = -\frac{\sqrt{3}}{3}.$$

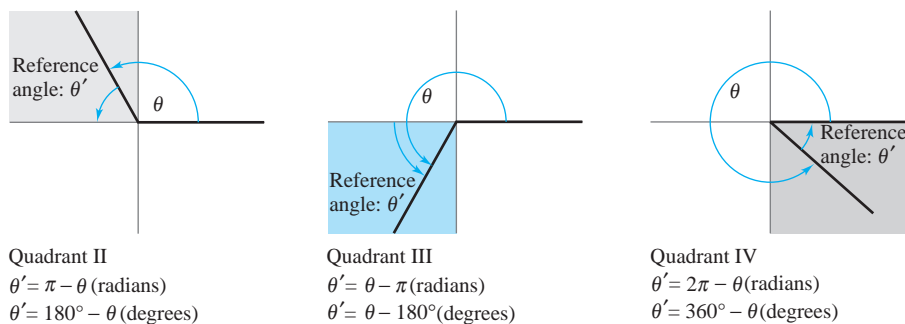


Figure D.35

EXAMPLE 3 Trigonometric Identities and Calculators

Evaluate the trigonometric expression.

- a. $\sin\left(-\frac{\pi}{3}\right)$ b. $\sec 60^\circ$ c. $\cos(1.2)$

Solution

- a. Using the reduction formula $\sin(-\theta) = -\sin \theta$, you can write

$$\sin\left(-\frac{\pi}{3}\right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$$

- b. Using the reciprocal identity $\sec \theta = 1/\cos \theta$, you can write

$$\sec 60^\circ = \frac{1}{\cos 60^\circ} = \frac{1}{1/2} = 2.$$

- c. Using a calculator, you can obtain

$$\cos(1.2) \approx 0.3624.$$

Remember that 1.2 is given in *radian* measure. Consequently, your calculator must be set in radian mode.

Solving Trigonometric Equations

How would you solve the equation $\sin \theta = 0$? You know that $\theta = 0$ is one solution, but this is not the only solution. Any one of the following values of θ is also a solution.

$$\dots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \dots$$

You can write this infinite solution set as $\{n\pi : n \text{ is an integer}\}$.

EXAMPLE 4 Solving a Trigonometric Equation

Solve the equation

$$\sin \theta = -\frac{\sqrt{3}}{2}.$$

Solution To solve the equation, you should consider that the sine is negative in Quadrants III and IV and that

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

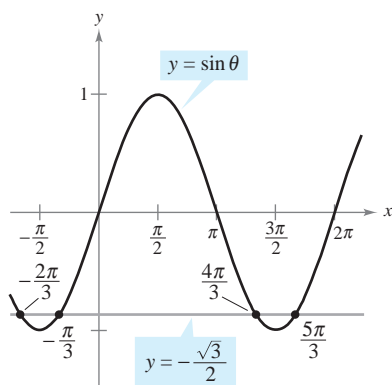
Thus, you are seeking values of θ in the third and fourth quadrants that have a reference angle of $\pi/3$. In the interval $[0, 2\pi]$, the two angles fitting these criteria are

$$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \quad \text{and} \quad \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

By adding integer multiples of 2π to each of these solutions, you obtain the following general solution.

$$\theta = \frac{4\pi}{3} + 2n\pi \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2n\pi, \quad \text{where } n \text{ is an integer.}$$

(See Figure D.36.)



Solution points of $\sin \theta = -\frac{\sqrt{3}}{2}$
Figure D.36

EXAMPLE 5 Solving a Trigonometric Equation

Solve $\cos 2\theta = 2 - 3 \sin \theta$, where $0 \leq \theta \leq 2\pi$.

Solution Using the double-angle identity $\cos 2\theta = 1 - 2 \sin^2 \theta$, you can rewrite the equation as follows.

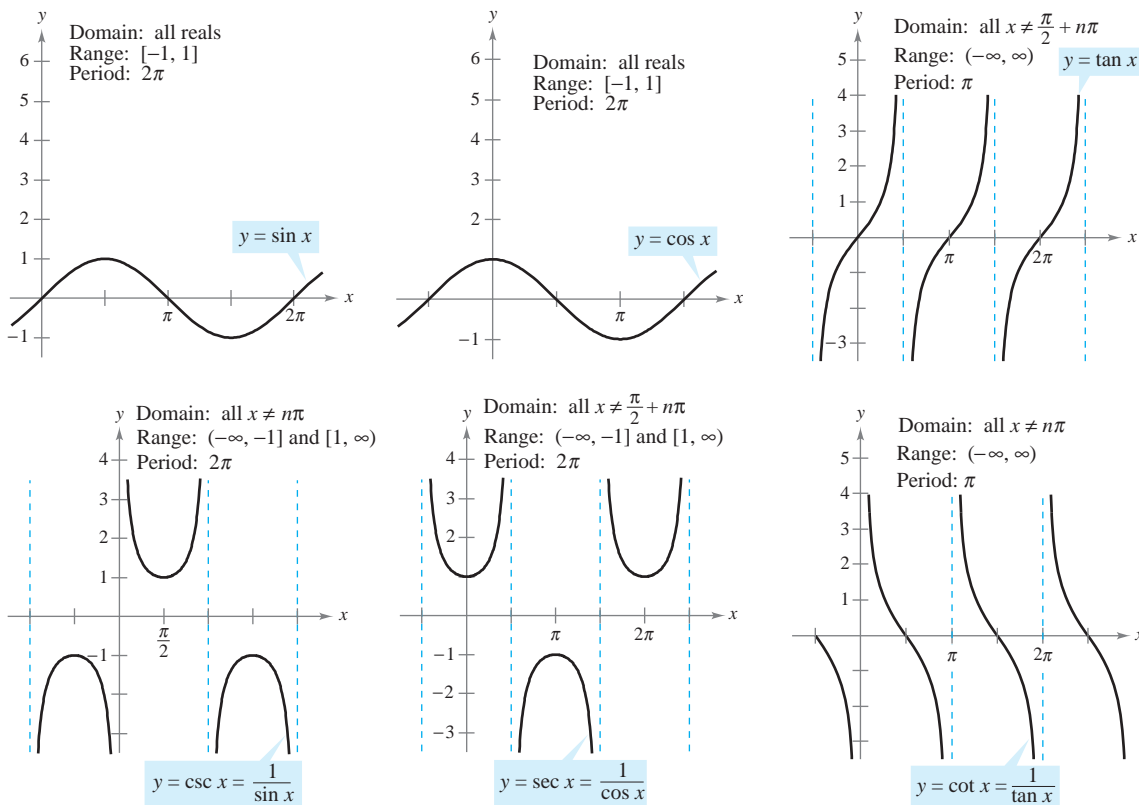
$$\begin{aligned} \cos 2\theta &= 2 - 3 \sin \theta && \text{Given equation} \\ 1 - 2 \sin^2 \theta &= 2 - 3 \sin \theta && \text{Trigonometric identity} \\ 0 &= 2 \sin^2 \theta - 3 \sin \theta + 1 && \text{Quadratic form} \\ 0 &= (2 \sin \theta - 1)(\sin \theta - 1) && \text{Factor.} \end{aligned}$$

If $2 \sin \theta - 1 = 0$, then $\sin \theta = 1/2$ and $\theta = \pi/6$ or $\theta = 5\pi/6$. If $\sin \theta - 1 = 0$, then $\sin \theta = 1$ and $\theta = \pi/2$. Thus, for $0 \leq \theta \leq 2\pi$, there are three solutions.

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \quad \text{or} \quad \frac{\pi}{2}$$

Graphs of Trigonometric Functions

A function f is **periodic** if there exists a nonzero number p such that $f(x + p) = f(x)$ for all x in the domain of f . The smallest such positive value of p (if it exists) is the **period** of f . The sine, cosine, secant, and cosecant functions each have a period of 2π , and the other two trigonometric functions have a period of π , as shown in Figure D.37.



The graphs of the six trigonometric functions
Figure D.37

Note in Figure D.37 that the maximum value of $\sin x$ and $\cos x$ is 1 and the minimum value is -1 . The graphs of the functions $y = a \sin bx$ and $y = a \cos bx$ oscillate between $-a$ and a , and hence have an **amplitude** of $|a|$. Furthermore, because $bx = 0$ when $x = 0$ and $bx = 2\pi$ when $x = 2\pi/b$, it follows that the functions $y = a \sin bx$ and $y = a \cos bx$ each have a period of $2\pi/|b|$. The table below summarizes the amplitudes and periods for some types of trigonometric functions.

Function	Period	Amplitude
$y = a \sin bx$ or $y = a \cos bx$	$\frac{2\pi}{ b }$	$ a $
$y = a \tan bx$ or $y = a \cot bx$	$\frac{\pi}{ b }$	Not applicable
$y = a \sec bx$ or $y = a \csc bx$	$\frac{2\pi}{ b }$	Not applicable

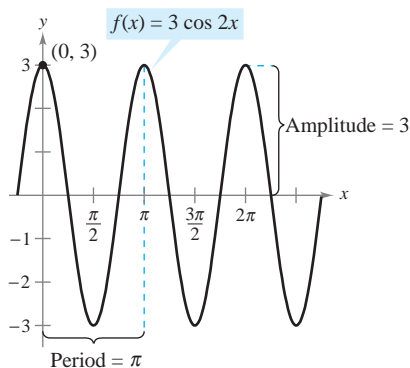


Figure D.38

EXAMPLE 6 Sketching the Graph of a Trigonometric Function

Sketch the graph of $f(x) = 3 \cos 2x$.

Solution The graph of $f(x) = 3 \cos 2x$ has an amplitude of 3 and a period of $2\pi/2 = \pi$. Using the basic shape of the graph of the cosine function, sketch one period of the function on the interval $[0, \pi]$, using the following pattern.

$$\text{Maximum: } (0, 3) \quad \text{Minimum: } \left(\frac{\pi}{2}, -3\right) \quad \text{Maximum: } (\pi, 3)$$

By continuing this pattern, you can sketch several cycles of the graph, as shown in Figure D.38.

Horizontal shifts, vertical shifts, and reflections can be applied to the graphs of trigonometric functions, as illustrated in Example 7.

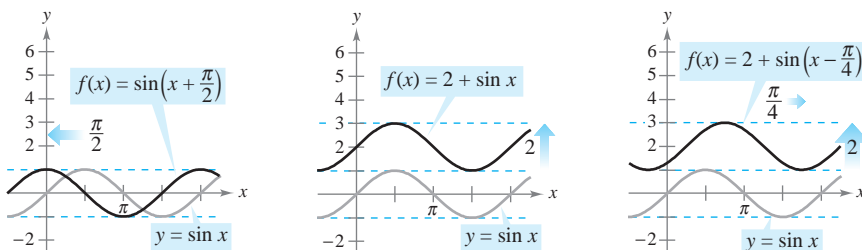
EXAMPLE 7 Shifts of Graphs of Trigonometric Functions

Sketch the graphs of the following functions.

a. $f(x) = \sin\left(x + \frac{\pi}{2}\right)$ b. $f(x) = 2 + \sin x$ c. $f(x) = 2 + \sin\left(x - \frac{\pi}{4}\right)$

Solution

- a. To sketch the graph of $f(x) = \sin(x + \pi/2)$, shift the graph of $y = \sin x$ to the left $\pi/2$ units, as shown in Figure D.39(a).
- b. To sketch the graph of $f(x) = 2 + \sin x$, shift the graph of $y = \sin x$ up two units, as shown in Figure D.39(b).
- c. To sketch the graph of $f(x) = 2 + \sin(x - \pi/4)$, shift the graph of $y = \sin x$ up two units and to the right $\pi/4$ units, as shown in Figure D.39(c).



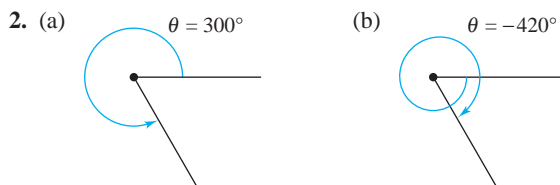
(a) Horizontal shift to the left (b) Vertical shift upward (c) Horizontal and vertical shifts

Transformations of the graph of $y = \sin x$

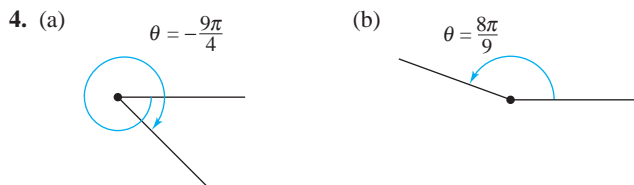
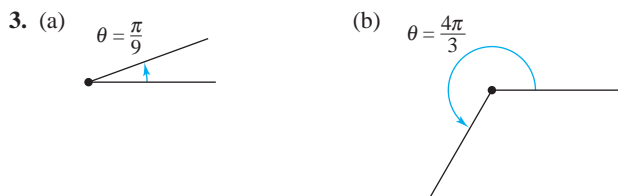
Figure D.39

EXERCISES FOR APPENDIX D.3

In Exercises 1 and 2, determine two coterminal angles (one positive and one negative) for each given angle. Express your answers in degrees.



In Exercises 3 and 4, determine two coterminal angles (one positive and one negative) for each given angle. Express your answers in radians.



In Exercises 5 and 6, express the angles in radian measure as multiples of π and as decimals accurate to three decimal places.

5. (a) 30° (b) 150° (c) 315° (d) 120°
 6. (a) -20° (b) -240° (c) -270° (d) 144°

In Exercises 7 and 8, express the angles in degree measure.

7. (a) $\frac{3\pi}{2}$ (b) $\frac{7\pi}{6}$ (c) $-\frac{7\pi}{12}$ (d) -2.367
 8. (a) $\frac{7\pi}{3}$ (b) $-\frac{11\pi}{30}$ (c) $\frac{11\pi}{6}$ (d) 0.438

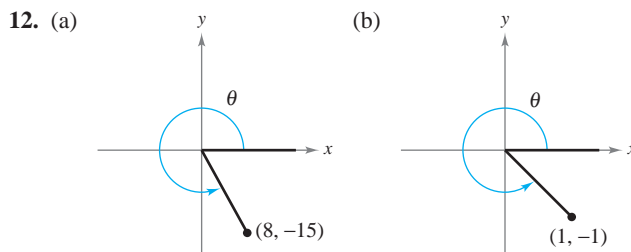
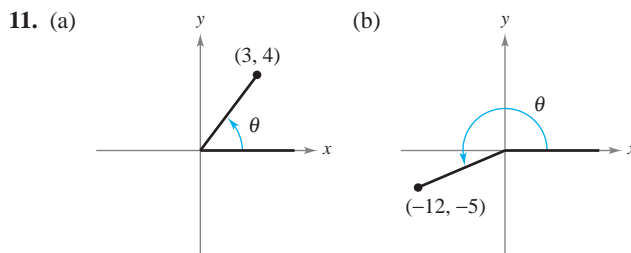
9. Let r represent the radius of a circle, θ the central angle (measured in radians), and s the length of the arc subtended by the angle. Use the relationship $s = r\theta$ to complete the table.

r	8 ft	15 in.	85 cm		
s	12 ft			96 in.	8642 mi
θ		1.6	$\frac{3\pi}{4}$	4	$\frac{2\pi}{3}$

10. **Angular Speed** A car is moving at the rate of 50 miles per hour, and the diameter of its wheels is 2.5 feet.

- (a) Find the number of revolutions per minute that the wheels are rotating.
 (b) Find the angular speed of the wheels in radians per minute.

In Exercises 11 and 12, determine all six trigonometric functions for the angle θ .

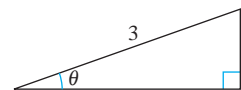
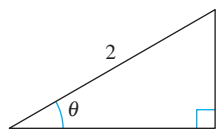


In Exercises 13 and 14, determine the quadrant in which θ lies.

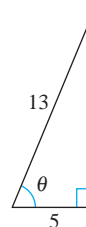
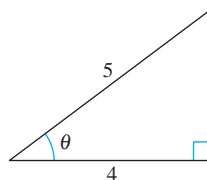
13. (a) $\sin \theta < 0$ and $\cos \theta < 0$
 (b) $\sec \theta > 0$ and $\cot \theta < 0$
 14. (a) $\sin \theta > 0$ and $\cos \theta < 0$
 (b) $\csc \theta < 0$ and $\tan \theta > 0$

In Exercises 15–18, evaluate the trigonometric function.

15. $\sin \theta = \frac{1}{2}$ 16. $\sin \theta = \frac{1}{3}$
 $\cos \theta = \square$ $\tan \theta = \square$



17. $\cos \theta = \frac{4}{5}$ 18. $\sec \theta = \frac{13}{5}$
 $\cot \theta = \square$ $\csc \theta = \square$



In Exercises 19–22, evaluate the sine, cosine, and tangent of each angle *without* using a calculator.

- | | |
|----------------------|----------------------|
| 19. (a) 60° | 20. (a) -30° |
| (b) 120° | (b) 150° |
| (c) $\frac{\pi}{4}$ | (c) $-\frac{\pi}{6}$ |
| (d) $\frac{5\pi}{4}$ | (d) $\frac{\pi}{2}$ |
- | | |
|-----------------------|-----------------------|
| 21. (a) 225° | 22. (a) 750° |
| (b) -225° | (b) 510° |
| (c) $\frac{5\pi}{3}$ | (c) $\frac{10\pi}{3}$ |
| (d) $\frac{11\pi}{6}$ | (d) $\frac{17\pi}{3}$ |

In Exercises 23–26, use a calculator to evaluate the trigonometric functions to four significant digits.

- | | |
|-------------------------|--------------------------|
| 23. (a) $\sin 10^\circ$ | 24. (a) $\sec 225^\circ$ |
| (b) $\csc 10^\circ$ | (b) $\sec 135^\circ$ |
- | | |
|------------------------------|----------------------|
| 25. (a) $\tan \frac{\pi}{9}$ | 26. (a) $\cot(1.35)$ |
| (b) $\tan \frac{10\pi}{9}$ | (b) $\tan(1.35)$ |

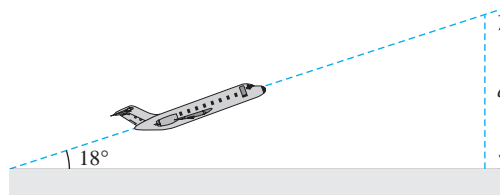
In Exercises 27–30, find two solutions of each equation. Express the results in radians ($0 \leq \theta < 2\pi$). Do not use a calculator.

- | | |
|--|---------------------------|
| 27. (a) $\cos \theta = \frac{\sqrt{2}}{2}$ | 28. (a) $\sec \theta = 2$ |
| (b) $\cos \theta = -\frac{\sqrt{2}}{2}$ | (b) $\sec \theta = -2$ |
- | | |
|-------------------------------|--|
| 29. (a) $\tan \theta = 1$ | 30. (a) $\sin \theta = \frac{\sqrt{3}}{2}$ |
| (b) $\cot \theta = -\sqrt{3}$ | (b) $\sin \theta = -\frac{\sqrt{3}}{2}$ |

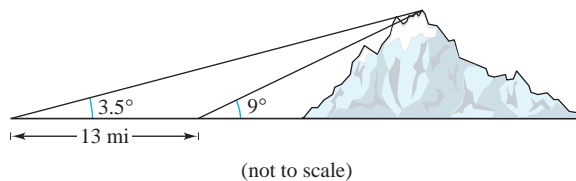
In Exercises 31–38, solve the equation for θ ($0 \leq \theta < 2\pi$).

- | | |
|---|---|
| 31. $2 \sin^2 \theta = 1$ | 32. $\tan^2 \theta = 3$ |
| 33. $\tan^2 \theta - \tan \theta = 0$ | 34. $2 \cos^2 \theta - \cos \theta = 1$ |
| 35. $\sec \theta \csc \theta = 2 \csc \theta$ | 36. $\sin \theta = \cos \theta$ |
| 37. $\cos^2 \theta + \sin \theta = 1$ | 38. $\cos \frac{\theta}{2} - \cos \theta = 1$ |

39. **Airplane Ascent** An airplane leaves the runway climbing at 18° with a speed of 275 feet per second (see figure). Find the altitude a of the plane after 1 minute.

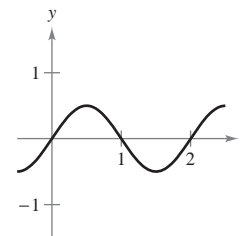
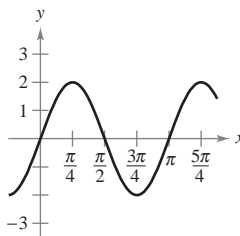


40. **Height of a Mountain** In traveling across flat land, you notice a mountain directly in front of you. Its angle of elevation (to the peak) is 3.5° . After you drive 13 miles closer to the mountain, the angle of elevation is 9° . Approximate the height of the mountain.

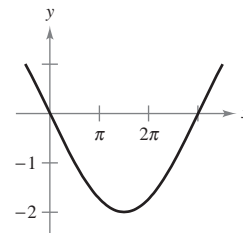
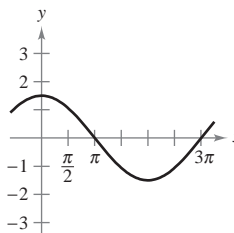


In Exercises 41–44, determine the period and amplitude of each function.

41. (a) $y = 2 \sin 2x$ (b) $y = \frac{1}{2} \sin \pi x$



42. (a) $y = \frac{3}{2} \cos \frac{x}{2}$ (b) $y = -2 \sin \frac{x}{3}$




43. $y = 3 \sin 4\pi x$

44. $y = \frac{2}{3} \cos \frac{\pi x}{10}$

In Exercises 45–48, find the period of the function.

45. $y = 5 \tan 2x$ 46. $y = 7 \tan 2\pi x$
 47. $y = \sec 5x$ 48. $y = \csc 4x$

 **Writing** In Exercises 49 and 50, use a graphing utility to graph each function f on the same set of coordinate axes for $c = -2$, $c = -1$, $c = 1$, and $c = 2$. Give a written description of the change in the graph caused by changing c .

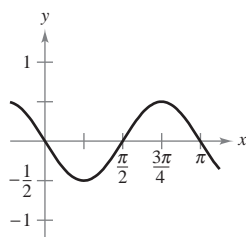
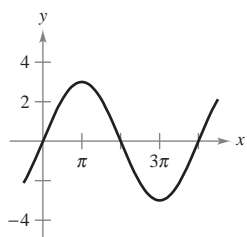
49. (a) $f(x) = c \sin x$ 50. (a) $f(x) = \sin x + c$
 (b) $f(x) = \cos(cx)$ (b) $f(x) = -\sin(2\pi x - c)$
 (c) $f(x) = \cos(\pi x - c)$ (c) $f(x) = c \cos x$

In Exercises 51–62, sketch the graph of the function.

51. $y = \sin \frac{x}{2}$ 52. $y = 2 \cos 2x$
 53. $y = -\sin \frac{2\pi x}{3}$ 54. $y = 2 \tan x$
 55. $y = \csc \frac{x}{2}$ 56. $y = \tan 2x$
 57. $y = 2 \sec 2x$ 58. $y = \csc 2\pi x$
 59. $y = \sin(x + \pi)$ 60. $y = \cos\left(x - \frac{\pi}{3}\right)$
 61. $y = 1 + \cos\left(x - \frac{\pi}{2}\right)$
 62. $y = 1 + \sin\left(x + \frac{\pi}{2}\right)$

Graphical Reasoning In Exercises 63 and 64, find a , b , and c such that the graph of the function matches the graph in the figure.

63. $y = a \cos(bx - c)$ 64. $y = a \sin(bx - c)$




65. **Think About It** Sketch the graphs of $f(x) = \sin x$, $g(x) = |\sin x|$, and $h(x) = \sin(|x|)$. In general, how are the graphs of $|f(x)|$ and $f(|x|)$ related to the graph of f ?

66. **Think About It** The model for the height h of a Ferris wheel car is

$$h = 51 + 50 \sin 8\pi t$$

where t is measured in minutes. (The Ferris wheel has a radius of 50 feet.) This model yields a height of 51 feet when $t = 0$. Alter the model so that the height of the car is 1 foot when $t = 0$.


 67. **Sales** Sales S , in thousands of units, of a seasonal product are modeled by

$$S = 58.3 + 32.5 \cos \frac{\pi t}{6}$$

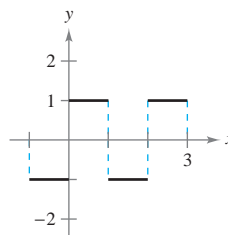
where t is the time in months (with $t = 1$ corresponding to January and $t = 12$ corresponding to December). Use a graphing utility to graph the model for S and determine the months when sales exceed 75,000 units.

68. **Investigation** Two trigonometric functions f and g have a period of 2, and their graphs intersect at $x = 5.35$.

- (a) Give one smaller and one larger positive value of x where the functions have the same value.
 (b) Determine one negative value of x where the graphs intersect.
 (c) Is it true that $f(13.35) = g(-4.65)$? Give a reason for your answer.

 **Pattern Recognition** In Exercises 69 and 70, use a graphing utility to compare the graph of f with the given graph. Try to improve the approximation by adding a term to $f(x)$. Use a graphing utility to verify that your new approximation is better than the original. Can you find other terms to add to make the approximation even better? What is the pattern? (In Exercise 69, sine terms can be used to improve the approximation and in Exercise 70, cosine terms can be used.)

69. $f(x) = \frac{4}{\pi} \left(\sin \pi x + \frac{1}{3} \sin 3\pi x \right)$



70. $f(x) = \frac{1}{2} - \frac{4}{\pi^2} \left(\cos \pi x + \frac{1}{9} \cos 3\pi x \right)$

