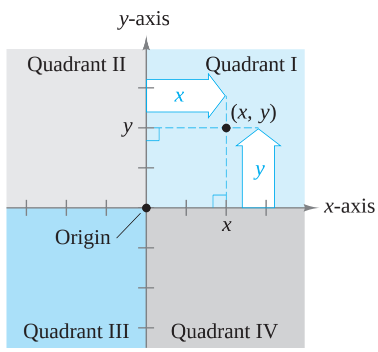
SECTION D.2 The Cartesian Plane

The Cartesian Plane • The Distance and Midpoint Formulas • Equations of Circles

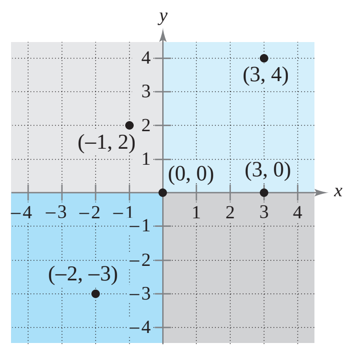
The Cartesian Plane

An **ordered pair** (x, y) of real numbers has x as its *first* member and y as its *second* member. The model for representing ordered pairs is called the **rectangular coordinate system,** or the **Cartesian plane,** after the French mathematician René Descartes. It is developed by considering two real lines intersecting at right angles (see Figure D.14).

The horizontal real line is usually called the ***x*-axis,** and the vertical real line is usually called the ***y*-axis.** Their point of intersection is the **origin.** The two axes divide the plane into four **quadrants.**



The Cartesian plane
Figure D.14



Points represented by ordered pairs
Figure D.15

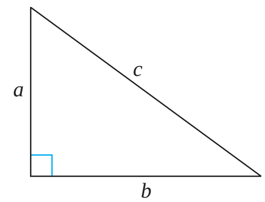
Each point in the plane is identified by an ordered pair (x, y) of real numbers x and y, called **coordinates** of the point. The number x represents the directed distance from the y-axis to the point, and the number y represents the directed distance from the x-axis to the point (see Figure A.14). For the point (x, y), the first coordinate is the x-coordinate or **abscissa,** and the second coordinate is the y-coordinate or **ordinate.** For example, Figure D.15 shows the locations of the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$ in the Cartesian plane.

NOTE The signs of the coordinates of a point determine the quadrant in which the point lies. For instance, if $x > 0$ and $y < 0$, then (x, y) lies in Quadrant IV.

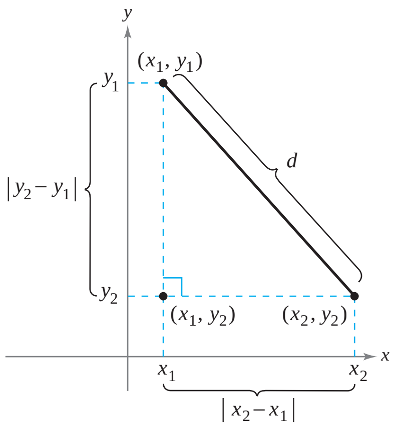
Note that an ordered pair (a, b) is used to denote either a point in the plane *or* an open interval on the real line. This, however, should not be confusing—the nature of the problem should clarify whether a point in the plane or an open interval is being discussed.

The Distance and Midpoint Formulas

Recall from the Pythagorean Theorem that, in a right triangle, the hypotenuse c and sides a and b are related by $a^2 + b^2 = c^2$. Conversely, if $a^2 + b^2 = c^2$, the triangle is a right triangle (see Figure D.16).



The Pythagorean Theorem:
$a^2 + b^2 = c^2$
Figure D.16



The distance between two points
Figure D.17

Suppose you want to determine the distance d between the two points (x_1, y_1) and (x_2, y_2) in the plane. If the points lie on a horizontal line, then $y_1 = y_2$ and the distance between the points is $|x_2 - x_1|$. If the points lie on a vertical line, then $x_1 = x_2$ and the distance between the points is $|y_2 - y_1|$. If the two points do not lie on a horizontal or vertical line, they can be used to form a right triangle, as shown in Figure D.17. The length of the vertical side of the triangle is $|y_2 - y_1|$, and the length of the horizontal side is $|x_2 - x_1|$. By the Pythagorean Theorem, it follows that

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}.$$

Replacing $|x_2 - x_1|^2$ and $|y_2 - y_1|^2$ by the equivalent expressions $(x_2 - x_1)^2$ and $(y_2 - y_1)^2$ produces the following result.

Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) in the plane is given by

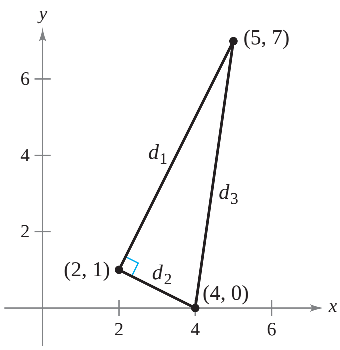
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 1 Finding the Distance Between Two Points

Find the distance between the points $(-2, 1)$ and $(3, 4)$.

Solution

$$\begin{aligned} d &= \sqrt{[3 - (-2)]^2 + (4 - 1)^2} && \text{Distance Formula} \\ &= \sqrt{(5)^2 + (3)^2} \\ &= \sqrt{25 + 9} \\ &= \sqrt{34} \\ &\approx 5.83 \end{aligned}$$



Verifying a right triangle
Figure D.18

EXAMPLE 2 Verifying a Right Triangle

Verify that the points (2, 1), (4, 0), and (5, 7) form the vertices of a right triangle.

Solution Figure D.18 shows the triangle formed by the three points. The lengths of the three sides are as follows.

$$d_1 = \sqrt{(5-2)^2 + (7-1)^2} = \sqrt{9+36} = \sqrt{45}$$
$$d_2 = \sqrt{(4-2)^2 + (0-1)^2} = \sqrt{4+1} = \sqrt{5}$$
$$d_3 = \sqrt{(5-4)^2 + (7-0)^2} = \sqrt{1+49} = \sqrt{50}$$

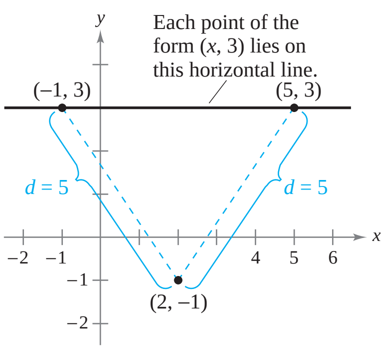
Because

$$d_1^2 + d_2^2 = 45 + 5 = 50 \qquad \text{Sum of squares of sides}$$

and

$$d_3^2 = 50 \qquad \text{Square of hypotenuse}$$

you can apply the Pythagorean Theorem to conclude that the triangle is a right triangle.



Given a distance, find a point.
Figure D.19

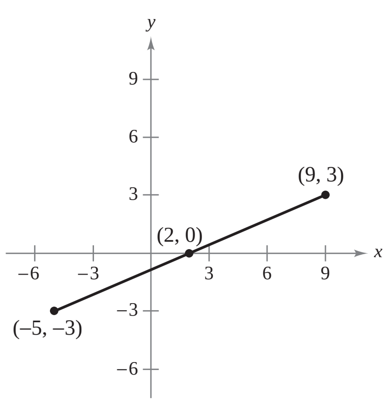
EXAMPLE 3 Using the Distance Formula

Find x such that the distance between $(x, 3)$ and $(2, -1)$ is 5.

Solution Using the Distance Formula, you can write the following.

$$5 = \sqrt{(x-2)^2 + [3-(-1)]^2} \qquad \text{Distance Formula}$$
$$25 = (x^2 - 4x + 4) + 16 \qquad \text{Square both sides.}$$
$$0 = x^2 - 4x - 5 \qquad \text{Write in standard form.}$$
$$0 = (x-5)(x+1) \qquad \text{Factor.}$$

Therefore, $x = 5$ or $x = -1$, and you can conclude that there are two solutions. That is, each of the points (5, 3) and $(-1, 3)$ lies five units from the point $(2, -1)$, as shown in Figure D.19.



Midpoint of a line segment
Figure D.20

The coordinates of the **midpoint** of the line segment joining two points can be found by "averaging" the x-coordinates of the two points and "averaging" the y-coordinates of the two points. That is, the midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) in the plane is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right). \qquad \text{Midpoint Formula}$$

For instance, the midpoint of the line segment joining the points $(-5, -3)$ and (9, 3) is

$$\left(\frac{-5+9}{2}, \frac{-3+3}{2}\right) = (2, 0)$$

as shown in Figure D.20.

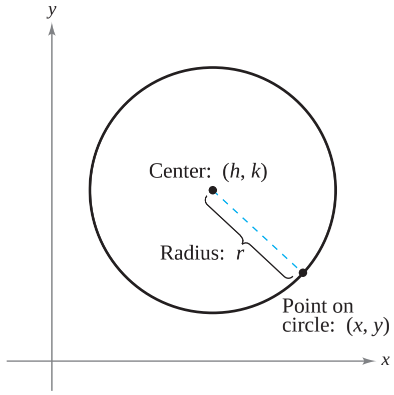
Equations of Circles

A **circle** can be defined as the set of all points in a plane that are equidistant from a fixed point. The fixed point is the **center** of the circle, and the distance between the center and a point on the circle is the **radius** (see Figure D.21).

You can use the Distance Formula to write an equation for the circle with center (h, k) and radius r. Let (x, y) be any point on the circle. Then the distance between (x, y) and the center (h, k) is given by

$$\sqrt{(x-h)^2 + (y-k)^2} = r.$$

By squaring both sides of this equation, you obtain the **standard form of the equation of a circle.**



Definition of a circle
Figure D.21

Standard Form of the Equation of a Circle

The point (x, y) lies on the circle of radius r and center (h, k) if and only if

$$(x-h)^2 + (y-k)^2 = r^2.$$

The standard form of the equation of a circle with center at the origin, $(h, k) = (0, 0)$, is

$$x^2 + y^2 = r^2.$$

If $r = 1$, the circle is called the **unit circle.**

EXAMPLE 4 Finding the Equation of a Circle

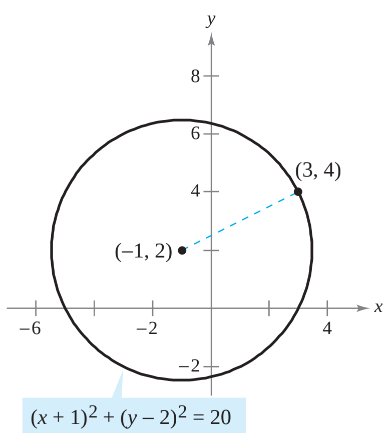
The point $(3, 4)$ lies on a circle whose center is at $(-1, 2)$, as shown in Figure D.22. Find an equation for the circle.

Solution The radius of the circle is the distance between $(-1, 2)$ and $(3, 4)$.

$$r = \sqrt{[3-(-1)]^2 + (4-2)^2} = \sqrt{16+4} = \sqrt{20}$$

You can write the standard form of the equation of this circle as

$$[x-(-1)]^2 + (y-2)^2 = \left(\sqrt{20}\right)^2$$
$$(x+1)^2 + (y-2)^2 = 20. \qquad \text{Standard form}$$



Standard form of the equation of a circle
Figure D.22

By squaring and simplifying, the equation $(x-h)^2 + (y-k)^2 = r^2$ can be written in the following **general form of the equation of a circle.**

$$Ax^2 + Ay^2 + Dx + Ey + F = 0, \quad A \neq 0$$

To convert such an equation to the standard form

$$(x-h)^2 + (y-k)^2 = p$$

you can use a process called **completing the square.** If $p > 0$, the graph of the equation is a circle. If $p = 0$, the graph is the single point (h, k). If $p < 0$, the equation has no graph.

EXAMPLE 5 Completing the Square

Sketch the graph of the circle whose general equation is

$$4x^2 + 4y^2 + 20x - 16y + 37 = 0.$$

Solution To complete the square, first divide by 4 so that the coefficients of x^2 and y^2 are both 1.

$$4x^2 + 4y^2 + 20x - 16y + 37 = 0 \qquad \text{General form}$$

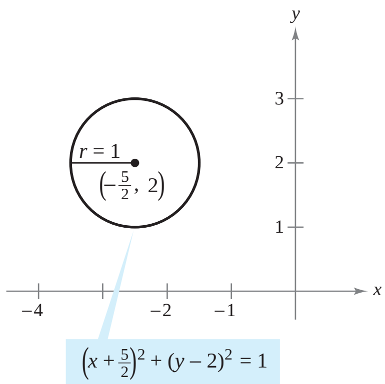
$$x^2 + y^2 + 5x - 4y + \frac{37}{4} = 0 \qquad \text{Divide by 4.}$$

$$(x^2 + 5x + \quad) + (y^2 - 4y + \quad) = -\frac{37}{4} \qquad \text{Group terms.}$$

$$\left(x^2 + 5x + \underbrace{\frac{25}{4}}_{(\text{half})^2}\right) + (y^2 - 4y + \underbrace{4}_{(\text{half})^2}) = -\frac{37}{4} + \frac{25}{4} + 4 \qquad \text{Complete the square by adding } \tfrac{25}{4} \text{ and 4 to both sides.}$$

$$\left(x + \frac{5}{2}\right)^2 + (y - 2)^2 = 1 \qquad \text{Standard form}$$

Note that you complete the square by adding the square of half the coefficient of x and the square of half the coefficient of y to both sides of the equation. The circle is centered at $\left(-\frac{5}{2}, 2\right)$ and its radius is 1, as shown in Figure D.23.



A circle with a radius of 1 and center at $\left(-\frac{5}{2}, 2\right)$
Figure D.23

We have now reviewed some fundamental concepts of *analytic geometry*. Because these concepts are in common use today, it is easy to overlook their revolutionary nature. At the time analytic geometry was being developed by Pierre de Fermat and René Descartes, the two major branches of mathematics—geometry and algebra—were largely independent of each other. Circles belonged to geometry and equations belonged to algebra. The coordination of the points on a circle and the solutions of an equation belongs to what is now called analytic geometry.

It is important to become skilled in analytic geometry so that you can move easily between geometry and algebra. For instance, in Example 4, you were given a geometric description of a circle and were asked to find an algebraic equation for the circle. Thus, you were moving from geometry to algebra. Similarly, in Example 5 you were given an algebraic equation and asked to sketch a geometric picture. In this case, you were moving from algebra to geometry. These two examples illustrate the two most common problems in analytic geometry.

1. Given a graph, find its equation.

Geometry ⇨ Algebra

2. Given an equation, find its graph.

Algebra ⇨ Geometry

In the next section, you will review other examples of these two types of problems.

EXERCISES FOR APPENDIX D.2

In Exercises 1–6, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

1. $(2, 1), (4, 5)$
2. $(-3, 2), (3, -2)$
3. $\left(\frac{1}{2}, 1\right), \left(-\frac{3}{2}, -5\right)$
4. $\left(\frac{2}{3}, -\frac{1}{3}\right), \left(\frac{5}{6}, 1\right)$
5. $\left(1, \sqrt{3}\right), (-1, 1)$
6. $(-2, 0), \left(0, \sqrt{2}\right)$

In Exercises 7–10, show that the points are the vertices of the polygon. (A rhombus is a quadrilateral whose sides are all of the same length.)

	Vertices	*Polygon*
7.	$(4, 0), (2, 1), (-1, -5)$	Right triangle
8.	$(1, -3), (3, 2), (-2, 4)$	Isosceles triangle
9.	$(0, 0), (1, 2), (2, 1), (3, 3)$	Rhombus
10.	$(0, 1), (3, 7), (4, 4), (1, -2)$	Parallelogram

In Exercises 11–14, determine the quadrant(s) in which (x, y) is located so that the condition(s) is (are) satisfied.

11. $x = -2$ and $y > 0$
12. $y < -2$
13. $xy > 0$
14. $(x, -y)$ is in the second quadrant.

15. ***Wal-Mart*** The number y of Wal-Mart stores for each year x from 1987 through 1996 is given in the table. *(Source: Wal-Mart Annual Report for 1996)*

x	1987	1988	1989	1990	1991
y	980	1114	1259	1399	1568

x	1992	1993	1994	1995	1996
y	1714	1848	1950	1985	1995

Select reasonable scales on the coordinate axes and plot the points (x, y).

16. ***Conjecture*** Plot the points $(2, 1)$, $(-3, 5)$, and $(7, -3)$ on a rectangular coordinate system. Then change the sign of the x-coordinate of each point and plot the three new points on the same rectangular coordinate system. What conjecture can you make about the location of a point when the sign of the x-coordinate is changed? Repeat the exercise for the case in which the sign of the y-coordinate is changed.

In Exercises 17–20, use the Distance Formula to determine whether the points lie on the same line.

17. $(0, -4), (2, 0), (3, 2)$
18. $(0, 4), (7, -6), (-5, 11)$
19. $(-2, 1), (-1, 0), (2, -2)$
20. $(-1, 1), (3, 3), (5, 5)$

In Exercises 21 and 22, find x such that the distance between the points is 5.

21. $(0, 0), (x, -4)$
22. $(2, -1), (x, 2)$

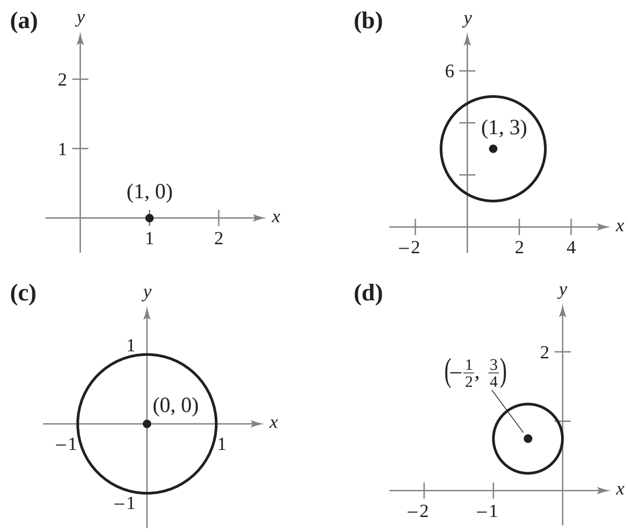
In Exercises 23 and 24, find y such that the distance between the points is 8.

23. $(0, 0), (3, y)$
24. $(5, 1), (5, y)$

25. Use the Midpoint Formula to find the three points that divide the line segment joining (x_1, y_1) and (x_2, y_2) into four equal parts.

26. Use the result of Exercise 25 to find the points that divide the line segment joining the given points into four equal parts.
(a) $(1, -2), (4, -1)$ (b) $(-2, -3), (0, 0)$

In Exercises 27–30, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]



27. $x^2 + y^2 = 1$
28. $(x - 1)^2 + (y - 3)^2 = 4$
29. $(x - 1)^2 + y^2 = 0$
30. $\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{3}{4}\right)^2 = \frac{1}{4}$

In Exercises 31–38, write the equation of the circle in general form.

31. Center: $(0, 0)$
Radius: 3

32. Center: $(0, 0)$
Radius: 5

33. Center: $(2, -1)$
Radius: 4

34. Center: $(-4, 3)$
Radius: $\frac{5}{8}$

35. Center: $(-1, 2)$
Point on circle: $(0, 0)$

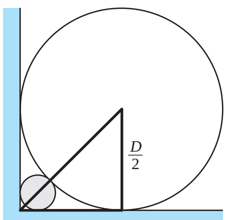
36. Center: $(3, -2)$
Point on circle: $(-1, 1)$

37. Endpoints of diameter: $(2, 5), (4, -1)$

38. Endpoints of diameter: $(1, 1), (-1, -1)$

39. ***Satellite Communication*** Write an equation for the path of a communications satellite in a circular orbit 22,000 miles above the earth. (Assume that the radius of the earth is 4000 miles.)

40. ***Building Design*** A circular air duct of diameter D is fit firmly into the right-angle corner where a basement wall meets the floor (see figure). Find the diameter of the largest water pipe that can be run in the right-angle corner behind the air duct.



In Exercises 41–48, write the equation of the circle in standard form and sketch its graph.

41. $x^2 + y^2 - 2x + 6y + 6 = 0$

42. $x^2 + y^2 - 2x + 6y - 15 = 0$

43. $x^2 + y^2 - 2x + 6y + 10 = 0$

44. $3x^2 + 3y^2 - 6y - 1 = 0$

45. $2x^2 + 2y^2 - 2x - 2y - 3 = 0$

46. $4x^2 + 4y^2 - 4x + 2y - 1 = 0$

47. $16x^2 + 16y^2 + 16x + 40y - 7 = 0$

48. $x^2 + y^2 - 4x + 2y + 3 = 0$

In Exercises 49 and 50, use a graphing utility to graph the equation. (*Hint:* It may be necessary to solve the equation for y and graph the resulting two equations.)

49. $4x^2 + 4y^2 - 4x + 24y - 63 = 0$

50. $x^2 + y^2 - 8x - 6y - 11 = 0$

In Exercises 51 and 52, sketch the set of all points satisfying the inequality. Use a graphing utility to verify your result.

51. $x^2 + y^2 - 4x + 2y + 1 \leq 0$

52. $(x - 1)^2 + \left(y - \frac{1}{2}\right)^2 > 1$

53. Prove that

$$\left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3}\right)$$

is one of the points of trisection of the line segment joining (x_1, y_1) and (x_2, y_2). Find the midpoint of the line segment joining

$$\left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3}\right)$$

and (x_2, y_2) to find the second point of trisection.

54. Use the results of Exercise 53 to find the points of trisection of the line segment joining the following points.

(a) $(1, -2), (4, 1)$ (b) $(-2, -3), (0, 0)$

True or False? **In Exercises 55–58, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

55. If $ab < 0$, the point (a, b) lies in either the second quadrant or the fourth quadrant.

56. The distance between the points $(a + b, a)$ and $(a - b, a)$ is $2b$.

57. If the distance between two points is zero, the two points must coincide.

58. If $ab = 0$, the point (a, b) lies on the x-axis or on the y-axis.

In Exercises 59–62, prove the statement.

59. The line segments joining the midpoints of the opposite sides of a quadrilateral bisect each other.

60. The perpendicular bisector of a chord of a circle passes through the center of the circle.

61. An angle inscribed in a semicircle is a right angle.

62. The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by use of appropriate technology.