

APPENDIX D

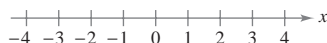
Precalculus Review

SECTION D.1 Real Numbers and the Real Line

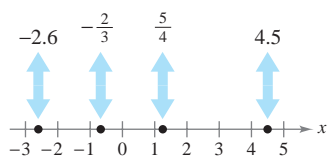
Real Numbers and the Real Line • Order and Inequalities • Absolute Value and Distance

Real Numbers and the Real Line

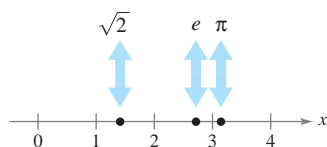
Real numbers can be represented by a coordinate system called the **real line** or x -axis (see Figure A.1). The real number corresponding to a point on the real line is the **coordinate** of the point. As Figure D.1 shows, it is customary to identify those points whose coordinates are integers.



The real line
Figure D.1



Rational numbers
Figure D.2



Irrational numbers
Figure D.3

The point on the real line corresponding to zero is the **origin** and is denoted by 0. The **positive direction** (to the right) is denoted by an arrowhead and is the direction of increasing values of x . Numbers to the right of the origin are **positive**. Numbers to the left of the origin are **negative**. The term **nonnegative** describes a number that is positive or zero. The term **nonpositive** describes a number that is negative or zero.

Each point on the real line corresponds to one and only one real number, and each real number corresponds to one and only one point on the real line. This type of relationship is called a **one-to-one-correspondence**.

Each of the four points in Figure D.2 corresponds to a **rational number**—one that can be expressed as the ratio of two integers. (Note that $4.5 = \frac{9}{2}$ and $-2.6 = \frac{-13}{5}$.) Rational numbers can be represented either by *terminating decimals* such as $\frac{2}{5} = 0.4$, or by *repeating decimals* such as $\frac{1}{3} = 0.333 \dots = 0.\bar{3}$.

Real numbers that are not rational are **irrational**. Irrational numbers cannot be represented as terminating or repeating decimals. In computations, irrational numbers are represented by decimal approximations. Here are three familiar examples.

$$\sqrt{2} \approx 1.414213562$$

$$\pi \approx 3.141592654$$

$$e \approx 2.718281828$$

(See Figure D.3.)

Order and Inequalities

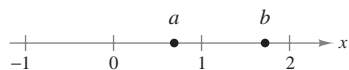
One important property of real numbers is that they can be **ordered**. If a and b are real numbers, a is **less than** b if $b - a$ is positive. This order is denoted by the **inequality**

$$a < b.$$

The statement “ b is **greater than** a ” is equivalent to saying that a is less than b . When three real numbers a , b , and c are ordered such that $a < b$ and $b < c$, we say that b is **between** a and c and $a < b < c$.

Geometrically, $a < b$ if and only if a lies to the *left* of b on the real line (see Figure D.4). For example, $1 < 2$ because 1 lies to the left of 2 on the real line.

The following properties are used in working with inequalities. Similar properties are obtained if $<$ is replaced by \leq and $>$ is replaced by \geq . (The symbols \leq and \geq mean **less than or equal to** and **greater than or equal to**, respectively.)



$a < b$ if and only if a lies to the left of b .

Figure D.4

Properties of Inequalities

Let a , b , c , d , and k be real numbers.

- | | |
|--|----------------------------------|
| 1. If $a < b$ and $b < c$, then $a < c$. | Transitive Property |
| 2. If $a < b$ and $c < d$, then $a + c < b + d$. | Add inequalities. |
| 3. If $a < b$, then $a + k < b + k$. | Add a constant. |
| 4. If $a < b$ and $k > 0$, then $ak < bk$. | Multiply by a positive constant. |
| 5. If $a < b$ and $k < 0$, then $ak > bk$. | Multiply by a negative constant. |

NOTE Note that you *reverse the inequality* when you multiply by a negative number. For example, if $x < 3$, then $-4x > -12$. This also applies to division by a negative number. Thus, if $-2x > 4$, then $x < -2$.

A **set** is a collection of elements. Two common sets are the set of real numbers and the set of points on the real line. Many problems in calculus involve **subsets** of one of these two sets. In such cases it is convenient to use **set notation** of the form $\{x: \text{condition on } x\}$, which is read as follows.

The set of all x such that a certain condition is true.

$$\{ \quad x \quad : \quad \text{condition on } x \}$$

For example, you can describe the set of positive real numbers as

$$\{x: x > 0\}. \quad \text{Set of positive real numbers}$$

Similarly, you can describe the set of nonnegative real numbers as

$$\{x: x \geq 0\}. \quad \text{Set of nonnegative real numbers}$$

The **union** of two sets A and B , denoted by $A \cup B$, is the set of elements that are members of A or B or both. The **intersection** of two sets A and B , denoted by $A \cap B$, is the set of elements that are members of A and B . Two sets are **disjoint** if they have no elements in common.

The most commonly used subsets are **intervals** on the real line. For example, the **open interval**

$$(a, b) = \{x: a < x < b\} \quad \text{Open interval}$$

is the set of all real numbers greater than a and less than b , where a and b are the **endpoints** of the interval. Note that the endpoints are not included in an open interval. Intervals that include their endpoints are **closed** and are denoted by

$$[a, b] = \{x: a \leq x \leq b\}. \quad \text{Closed interval}$$

The nine basic types of intervals on the real line are shown in the table below. The first four are **bounded intervals** and the remaining five are **unbounded intervals**. Unbounded intervals are also classified as open or closed. The intervals $(-\infty, b)$ and (a, ∞) are open, the intervals $(-\infty, b]$ and $[a, \infty)$ are closed, and the interval $(-\infty, \infty)$ is considered to be both open *and* closed.

Intervals on the Real Line

	Interval Notation	Set Notation	Graph
Bounded open interval	(a, b)	$\{x: a < x < b\}$	
Bounded closed interval	$[a, b]$	$\{x: a \leq x \leq b\}$	
Bounded intervals (neither open nor closed)	$[a, b)$	$\{x: a \leq x < b\}$	
	$(a, b]$	$\{x: a < x \leq b\}$	
Unbounded open intervals	$(-\infty, b)$	$\{x: x < b\}$	
	(a, ∞)	$\{x: x > a\}$	
Unbounded closed intervals	$(-\infty, b]$	$\{x: x \leq b\}$	
	$[a, \infty)$	$\{x: x \geq a\}$	
Entire real line	$(-\infty, \infty)$	$\{x: x \text{ is a real number}\}$	

NOTE The symbols ∞ and $-\infty$ refer to positive and negative infinity. These symbols do not denote real numbers. They simply enable you to describe unbounded conditions more concisely. For instance, the interval $[a, \infty)$ is unbounded to the right because it includes *all* real numbers that are greater than or equal to a .

EXAMPLE 1 Liquid and Gaseous States of Water

Describe the intervals on the real line that correspond to the temperature x (in degrees Celsius) for water in

- a. a liquid state b. a gaseous state.

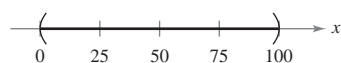
Solution

- a. Water is in a liquid state at temperatures greater than 0° and less than 100° , as shown in Figure D.5(a).

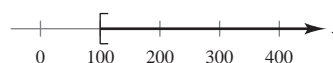
$$(0, 100) = \{x: 0 < x < 100\}$$

- b. Water is in a gaseous state (steam) at temperatures greater than or equal to 100° , as shown in Figure D.5(b).

$$[100, \infty) = \{x: x \geq 100\}$$



(a) Temperature range of water
(in degrees Celsius)



(b) Temperature range of steam
(in degrees Celsius)

Figure D.5

A real number a is a **solution** of an inequality if the inequality is **satisfied** (is true) when a is substituted for x . The set of all solutions is the **solution set** of the inequality.

EXAMPLE 2 Solving an Inequality

Solve $2x - 5 < 7$.

Solution

$$2x - 5 < 7$$

Original inequality

$$2x - 5 + 5 < 7 + 5$$

Add 5 to both sides.

$$2x < 12$$

Simplify.

$$\frac{1}{2}(2x) < \frac{1}{2}(12)$$

Multiply both sides by $\frac{1}{2}$.

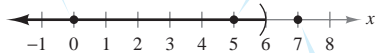
$$x < 6$$

Simplify.

The solution set is $(-\infty, 6)$.

If $x = 0$, then $2(0) - 5 = -5 < 7$.

If $x = 5$, then $2(5) - 5 = 5 < 7$.



If $x = 7$, then $2(7) - 5 = 9 > 7$.

Checking solutions of $2x - 5 < 7$

Figure D.6

NOTE In Example 2, all five inequalities listed as steps in the solution are called **equivalent** because they have the same solution set.

Once you have solved an inequality, check some x -values in your solution set to verify that they satisfy the original inequality. You should also check some values outside your solution set to verify that they *do not* satisfy the inequality. For example, Figure D.6 shows that when $x = 0$ or $x = 5$ the inequality $2x - 5 < 7$ is satisfied, but when $x = 7$ the inequality $2x - 5 < 7$ is not satisfied.

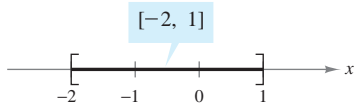
EXAMPLE 3 Solving a Double Inequality

Solve $-3 \leq 2 - 5x \leq 12$.

Solution

$-3 \leq 2 - 5x \leq 12$	Original inequality
$-3 - 2 \leq 2 - 5x - 2 \leq 12 - 2$	Subtract 2.
$-5 \leq -5x \leq 10$	Simplify.
$\frac{-5}{-5} \geq \frac{-5x}{-5} \geq \frac{10}{-5}$	Divide by -5 and reverse both inequalities.
$1 \geq x \geq -2$	Simplify.

The solution set is $[-2, 1]$, as shown in Figure D.7.



Solution set of $-3 \leq 2 - 5x \leq 12$

Figure D.7

The inequalities in Examples 2 and 3 are **linear inequalities**—that is, they involve first-degree polynomials. To solve inequalities involving polynomials of higher degree, use the fact that a polynomial can change signs *only* at its real **zeros** (the numbers that make the polynomial zero). Between two consecutive real zeros, a polynomial must be either entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real line into **test intervals** in which the polynomial has no sign changes. Thus, if a polynomial has the factored form

$$(x - r_1)(x - r_2) \cdots (x - r_n), \quad r_1 < r_2 < r_3 < \cdots < r_n$$

the test intervals are

$$(-\infty, r_1), (r_1, r_2), \dots, (r_{n-1}, r_n), \text{ and } (r_n, \infty).$$

To determine the sign of the polynomial in each test interval, you need to test only *one value* from the interval.

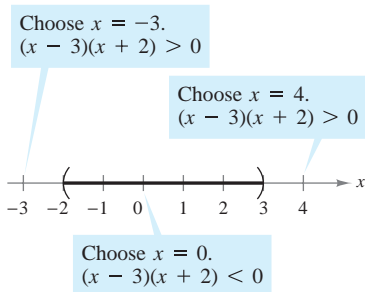
EXAMPLE 4 Solving a Quadratic Inequality

Solve $x^2 < x + 6$.

Solution

$x^2 < x + 6$	Original inequality
$x^2 - x - 6 < 0$	Write in standard form.
$(x - 3)(x + 2) < 0$	Factor.

The polynomial $x^2 - x - 6$ has $x = -2$ and $x = 3$ as its zeros. Thus, you can solve the inequality by testing the sign of $x^2 - x - 6$ in each of the test intervals $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$. To test an interval, choose any number in the interval and compute the sign of $x^2 - x - 6$. After doing this, you will find that the polynomial is positive for all real numbers in the first and third intervals and negative for all real numbers in the second interval. The solution of the original inequality is therefore $(-2, 3)$, as shown in Figure D.8.



Testing an interval

Figure D.8

Absolute Value and Distance

If a is a real number, the **absolute value** of a is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$$

The absolute value of a number cannot be negative. For example, let $a = -4$. Then, because $-4 < 0$, you have

$$|a| = |-4| = -(-4) = 4.$$

Remember that the symbol $-a$ does not necessarily mean that $-a$ is negative.

Operations with Absolute Value

Let a and b be real numbers and let n be a positive integer.

1. $|ab| = |a| |b|$
2. $\frac{|a|}{|b|} = \frac{|a|}{|b|}, \quad b \neq 0$
3. $|a| = \sqrt{a^2}$
4. $|a^n| = |a|^n$

NOTE You are asked to prove these properties in Exercises 73, 75, 76, and 77.

Properties of Inequalities and Absolute Value

Let a and b be real numbers and let k be a positive real number.

1. $-|a| \leq a \leq |a|$
2. $|a| \leq k$ if and only if $-k \leq a \leq k$.
3. $k \leq |a|$ if and only if $k \leq a$ or $a \leq -k$.
4. *Triangle Inequality:* $|a + b| \leq |a| + |b|$

Properties 2 and 3 are also true if \leq is replaced by $<$.

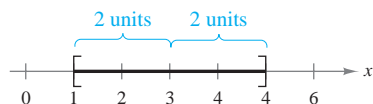
EXAMPLE 5 Solving an Absolute Value Inequality

Solve $|x - 3| \leq 2$.

Solution Using the second property of inequalities and absolute value, you can rewrite the original inequality as a double inequality.

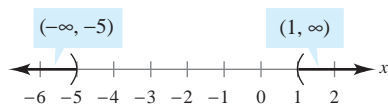
$$\begin{aligned} -2 &\leq x - 3 \leq 2 && \text{Write as double inequality.} \\ -2 + 3 &\leq x - 3 + 3 \leq 2 + 3 && \text{Add 3.} \\ 1 &\leq x \leq 5 && \text{Simplify.} \end{aligned}$$

The solution set is $[1, 5]$, as shown in Figure D.9.



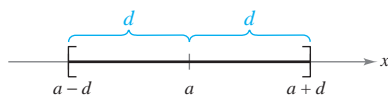
Solution set of $|x - 3| \leq 2$

Figure D.9

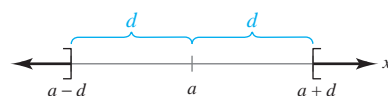


Solution set of $3 < |x + 2|$

Figure D.10



Solution set of $|x - a| \leq d$



Solution set of $|x - a| \geq d$

Figure D.11

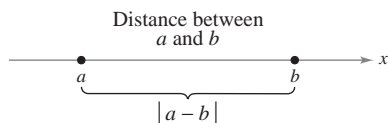


Figure D.12

EXAMPLE 6 A Two-Interval Solution Set

Solve $3 < |x + 2|$.

Solution Using the third property of inequalities and absolute value, you can rewrite the original inequality as two linear inequalities.

$$3 < x + 2 \quad \text{or} \quad x + 2 < -3$$

$$1 < x \quad \text{or} \quad x < -5$$

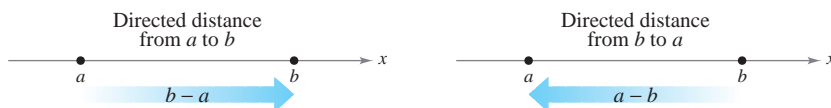
The solution set is the union of the disjoint intervals $(-\infty, -5)$ and $(1, \infty)$, as shown in Figure D.10.

Examples 5 and 6 illustrate the general results shown in Figure D.11. Note that if $d > 0$, the solution set for the inequality $|x - a| \leq d$ is a *single* interval, whereas the solution set for the inequality $|x - a| \geq d$ is the union of *two* disjoint intervals.

The **distance between two points** a and b on the real line is given by

$$d = |a - b| = |b - a|.$$

The **directed distance from a to b** is $b - a$ and the **directed distance from b to a** is $a - b$, as shown in Figure D.12.



EXAMPLE 7 Distance on the Real Line

a. The distance between -3 and 4 is

$$|4 - (-3)| = |7| = 7 \quad \text{or} \quad |-3 - 4| = |-7| = 7.$$

(See Figure D.13.)

b. The directed distance from -3 to 4 is $4 - (-3) = 7$.

c. The directed distance from 4 to -3 is $-3 - 4 = -7$.

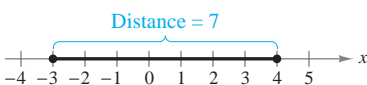


Figure D.13

The **midpoint** of an interval with endpoints a and b is the average value of a and b . That is,

$$\text{Midpoint of interval } (a, b) = \frac{a + b}{2}.$$

To show that this is the midpoint, you need only show that $(a + b)/2$ is equidistant from a and b .

EXERCISES FOR APPENDIX D.1

In Exercises 1–10, determine whether the real number is rational or irrational.

- 1. 0.7
- 2. -3678
- 3. $\frac{3\pi}{2}$
- 4. $3\sqrt{2} - 1$
- 5. $4.\overline{3451}$
- 6. $\frac{22}{7}$
- 7. $\sqrt[3]{64}$
- 8. $0.\overline{8177}$
- 9. $4\frac{5}{8}$
- 10. $(\sqrt{2})^3$

In Exercises 11–14, express the repeating decimal as a ratio of integers using the following procedure. If $x = 0.6363\dots$, then $100x = 63.6363\dots$. Subtracting the first equation from the second produces $99x = 63$ or $x = \frac{63}{99} = \frac{7}{11}$.

- 11. $0.\overline{36}$
- 12. $0.\overline{318}$
- 13. $0.\overline{297}$
- 14. $0.\overline{9900}$

15. Given $a < b$, determine which of the following are true.

- (a) $a + 2 < b + 2$
- (b) $5b < 5a$
- (c) $5 - a > 5 - b$
- (d) $\frac{1}{a} < \frac{1}{b}$
- (e) $(a - b)(b - a) > 0$
- (f) $a^2 < b^2$

16. Complete the table with the appropriate interval notation, set notation, and graph on the real line.

Interval Notation	Set Notation	Graph
$(-\infty, -4]$		
	$\{x: 3 \leq x \leq \frac{11}{2}\}$	
$(-1, 7)$		

In Exercises 17–20, verbally describe the subset of real numbers represented by the inequality. Sketch the subset on the real number line, and state whether the interval is bounded or unbounded.

- 17. $-3 < x < 3$
- 18. $x \geq 4$
- 19. $x \leq 5$
- 20. $0 \leq x < 8$

In Exercises 21–24, use inequality and interval notation to describe the set.

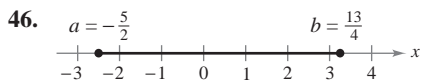
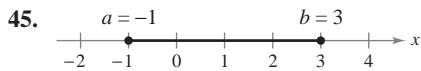
21. y is at least 4.

- 22. q is nonnegative.
- 23. The interest rate on loans is expected to be greater than 3% and no more than 7%.
- 24. The temperature T is forecast to be above 90° today.

In Exercises 25–44, solve the inequality and graph the solution on the real line.

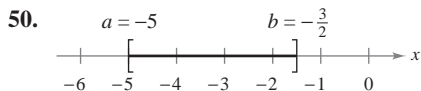
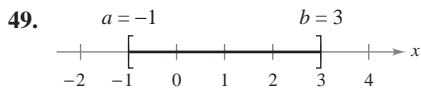
- 25. $2x - 1 \geq 0$
- 26. $3x + 1 \geq 2x + 2$
- 27. $-4 < 2x - 3 < 4$
- 28. $0 \leq x + 3 < 5$
- 29. $\frac{x}{2} + \frac{x}{3} > 5$
- 30. $x > \frac{1}{x}$
- 31. $|x| < 1$
- 32. $\frac{x}{2} - \frac{x}{3} > 5$
- 33. $\left|\frac{x-3}{2}\right| \geq 5$
- 34. $\left|\frac{x}{2}\right| > 3$
- 35. $|x - a| < b, b > 0$
- 36. $|x + 2| < 5$
- 37. $|2x + 1| < 5$
- 38. $|3x + 1| \geq 4$
- 39. $\left|1 - \frac{2}{3}x\right| < 1$
- 40. $|9 - 2x| < 1$
- 41. $x^2 \leq 3 - 2x$
- 42. $x^4 - x \leq 0$
- 43. $x^2 + x - 1 \leq 5$
- 44. $2x^2 + 1 < 9x - 3$

In Exercises 45–48, find the directed distance from a to b , the directed distance from b to a , and the distance between a and b .



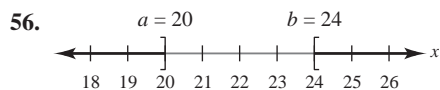
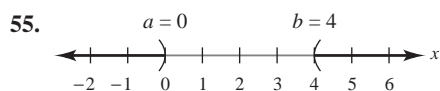
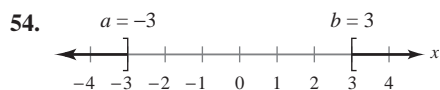
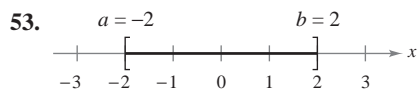
- 47. (a) $a = 126, b = 75$
- (b) $a = -126, b = -75$
- 48. (a) $a = 9.34, b = -5.65$
- (b) $a = \frac{16}{5}, b = \frac{112}{75}$

In Exercises 49–52, find the midpoint of the interval.



- 51. (a) $[7, 21]$
- (b) $[8.6, 11.4]$
- 52. (a) $[-6.85, 9.35]$
- (b) $[-4.6, -1.3]$

In Exercises 53–58, use absolute values to define the interval or pair of intervals on the real line.



57. (a) All numbers that are at most ten units from 12.
 (b) All numbers that are at least ten units from 12.

58. (a) y is at most two units from a .
 (b) y is less than δ units from c .

59. **Profit** The revenue from selling x units of a product is

$$R = 115.95x$$

and the cost of producing x units is

$$C = 95x + 750.$$

To make a (positive) profit, R must be greater than C . For what values of x will the product return a profit?

60. **Fleet Costs** A utility company has a fleet of vans. The annual operating cost of each van is estimated to be

$$C = 0.32m + 2300$$

where C is measured in dollars and m is measured in miles. The company wants the annual operating cost of each van to be less than \$10,000. To do this, m must be less than what value?

61. **Fair Coin** To determine whether a coin is fair (has an equal probability of landing tails up or heads up), an experimenter tosses it 100 times and records the number of heads x . The coin is declared unfair if

$$\left| \frac{x - 50}{5} \right| \geq 1.645.$$

For what values of x will the coin be declared unfair?

62. **Daily Production** The estimated daily production p at a refinery is

$$|p - 2,250,000| < 125,000$$

where p is measured in barrels of oil. Determine the high and low production levels.

In Exercises 63 and 64, determine which of the two real numbers is greater.

63. (a) π or $\frac{355}{113}$ 64. (a) $\frac{224}{151}$ or $\frac{144}{97}$
 (b) π or $\frac{22}{7}$ (b) $\frac{73}{81}$ or $\frac{6427}{7132}$

65. **Approximation—Powers of 10** The speed of light is 2.998×10^8 meters per second. Which best estimates the distance in meters that light travels in a year?

- (a) 9.5×10^5 (b) 9.5×10^{15}
 (c) 9.5×10^{12} (d) 9.6×10^{16}

66. **Writing** The accuracy of an approximation to a number is related to how many significant digits there are in the approximation. Write a definition for significant digits and illustrate the concept with examples.

True or False? In Exercises 67–72, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

67. The reciprocal of a nonzero integer is an integer.
 68. The reciprocal of a nonzero rational number is a rational number.
 69. Each real number is either rational or irrational.
 70. The absolute value of each real number is positive.
 71. If $x < 0$, then $\sqrt{x^2} = -x$.
 72. If a and b are any two distinct real numbers, then $a < b$ or $a > b$.

In Exercises 73–80, prove the property.

73. $|ab| = |a||b|$
 74. $|a - b| = |b - a|$
 [Hint: $(a - b) = (-1)(b - a)$]
 75. $\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$, $b \neq 0$
 76. $|a| = \sqrt{a^2}$
 77. $|a^n| = |a|^n$, $n = 1, 2, 3, \dots$
 78. $-|a| \leq a \leq |a|$
 79. $|a| \leq k$, if and only if $-k \leq a \leq k$, $k > 0$.
 80. $k \leq |a|$ if and only if $k \leq a$ or $a \leq -k$, $k > 0$.

81. Find an example for which $|a - b| > |a| - |b|$, and an example for which $|a - b| = |a| - |b|$. Then prove that $|a - b| \geq |a| - |b|$ for all a, b .

82. Show that the maximum of two numbers a and b is given by the formula

$$\max(a, b) = \frac{1}{2}(a + b + |a - b|).$$

Derive a similar formula for $\min(a, b)$.